

INITIAL COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF MEROMORPHIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we extend the concept of bi-univalent to the class of meromorphic functions. We propose to investigate the coefficient estimates for two classes of meromorphic bi-univalent functions. Also, we find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in these new classes. Some interesting remarks and applications of the results presented here are also discussed.

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1. INTRODUCTION

Let \mathcal{A} denote the class of functions of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by \mathcal{S} we shall denote the class of all functions in \mathcal{A} which are univalent in \mathbb{U} .

It is well known that every function $h \in \mathcal{S}$ has an inverse h^{-1} , defined by

$$h^{-1}(h(z)) = z, \quad (z \in \mathbb{U})$$

and

$$h(h^{-1}(w)) = w, \quad (|w| < r_0(h); \quad r_0(h) \geq \frac{1}{4}),$$

where

$$h^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function $h \in \mathcal{S}$ is said to be bi-univalent in \mathbb{U} if both $h(z)$ and $h^{-1}(z)$ are univalent in \mathbb{U} . Let $\Sigma_{\mathcal{B}}$ denote the class of bi-univalent functions in \mathbb{U} given by (1.1).

In 1967, Lewin [14] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. On the other hand, Brannan and Clunie [2] (see also [3, 4, 23]) and Netanyahu [15] made an attempt to introduce various subclasses of the bi-univalent function class $\Sigma_{\mathcal{B}}$ and obtained non-sharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} := \{1, 2, 3, \dots\}$) is still an open problem. Following Brannan and Taha [4], many researchers (see [1, 5, 7, 8, 10, 16, 20, 22, 24, 26]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class $\Sigma_{\mathcal{B}}$ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Let Σ denote the class of functions f of the form

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \quad (1.3)$$

which are meromorphic univalent functions defined in

$$\mathcal{V} := \{z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}.$$

It is well known that every function $f \in \Sigma$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{V})$$

and

$$f^{-1}(f(w)) = w \quad (M < |w| < \infty, M > 0).$$

Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \quad (1.4)$$

where $M < |w| < \infty$.

The coefficient problem was investigated for various interesting subclasses of the meromorphic univalent functions (see, for example [6, 12, 18]). In 1951, Springer [21] conjectured on the coefficient of the inverse of meromorphic univalent functions, latter the problem was investigated by many researchers for various subclasses (see, for details [11, 12, 13, 19, 25]).

Analogous to the bi-univalent analytic functions, a function $f \in \Sigma$ is said to be meromorphic bi-univalent if both f and f^{-1} are meromorphic univalent in \mathcal{V} . We denote by $\Sigma_{\mathcal{M}}$ the class of all meromorphic bi-univalent functions in \mathcal{V} given by (1.3).

A function f in the class Σ is said to be meromorphic bi-univalent starlike of order α ($0 \leq \alpha < 1$) if it satisfies the following inequalities

$$f \in \Sigma_{\mathcal{M}}, \quad \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{V}) \text{ and } \Re \left(\frac{wg'(w)}{g(w)} \right) > \alpha \quad (w \in \mathcal{V}),$$

where $g(w) = f^{-1}(w)$ is the inverse of $f(z)$ whose series expansion is given by (1.4), a simple calculation shows that

$$g(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0b_1}{w^2} - \frac{b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2}{w^3} + \dots \quad (1.5)$$

We denote by $\Sigma_{\mathcal{M}}^*(\alpha)$ the class of all meromorphic bi-univalent starlike functions of order α . Similarly, a function f in the class Σ is said to be meromorphic bi-univalent strongly starlike of order α ($0 < \alpha \leq 1$) if it satisfies the following conditions

$$f \in \Sigma_{\mathcal{M}}, \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathcal{V}) \text{ and } \left| \arg \left(\frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (w \in \mathcal{V}),$$

where $g(w)$ is given by (1.5). We denote by $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha)$ the class of all meromorphic bi-univalent strongly starlike functions of order α . The classes $\Sigma_{\mathcal{M}}^*(\alpha)$ and $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha)$ were introduced and studied by Halim et al. [9].

Motivated by the works of Halim et al. [9] we define the following general subclasses $\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ and $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ of the function class Σ .

Definition 1.1. A function f given by (1.3) is said to be in the class $\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_{\mathcal{M}}, \quad \Re \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) > \alpha \quad (\mu \geq 0, \lambda \geq 1; z \in \mathcal{V}) \quad (1.6)$$

and

$$\Re \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right) > \alpha \quad (\mu \geq 0, \lambda \geq 1; w \in \mathcal{V}) \quad (1.7)$$

for some $\alpha (0 \leq \alpha < 1)$, where g is given by (1.5).

Definition 1.2. A function f given by (1.3) is said to be in the class $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma_{\mathcal{M}}, \quad \left| \arg \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2} \quad (\mu \geq 0, \lambda \geq 1; z \in \mathcal{V}) \quad (1.8)$$

and

$$\left| \arg \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2} \quad (\mu \geq 0, \lambda \geq 1; w \in \mathcal{V}) \quad (1.9)$$

for some $\alpha (0 < \alpha \leq 1)$, where g is given by (1.5).

It is interesting to note that, for $\lambda = 1$ and $\mu = 0$ the classes $\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ and $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ respectively, reduces to the classes $\Sigma_{\mathcal{M}}^*(\alpha)$ and $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha)$ introduced and studied by Halim et al. [9].

The object of the present paper is to extend the concept of bi-univalent to the class of meromorphic functions defined on \mathcal{V} and find estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the above-defined classes $\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ and $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ of the function class $\Sigma_{\mathcal{M}}$ by employing the techniques used earlier by Halim et al. [9].

In order to derive our main results, we shall need the following lemma.

Lemma 1.3. (see [17]) *If $\varphi \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where \mathcal{P} is the family of all functions φ , analytic in \mathbb{U} , for which*

$$\Re\{\varphi(z)\} > 0 \quad (z \in \mathbb{U}),$$

where

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASSES $\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda)$ AND $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda)$

We begin this section by finding the estimates on the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda)$.

Theorem 2.1. *Let the function $f(z)$ given by (1.3) be in the following class:*

$$\Sigma_{\mathcal{M}}^*(\alpha, \mu, \lambda) \quad (0 \leq \alpha < 1; \lambda \geq 1; \mu \geq 0).$$

Then

$$|b_0| \leq \frac{2(1-\alpha)}{\lambda-\mu} \quad (2.1)$$

and

$$|b_1| \leq 2(1-\alpha) \sqrt{\frac{(1-\mu)^2(1-\alpha)^2}{(\lambda-\mu)^4} + \frac{1}{(2\lambda-\mu)^2}}. \quad (2.2)$$

Proof. It follows from (1.6) and (1.7) that

$$(1-\lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = \alpha + (1-\alpha)p(z) \quad (2.3)$$

and

$$(1-\lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = \alpha + (1-\alpha)q(w), \quad (2.4)$$

where $p(z)$ and $q(w)$ are functions with positive real part in \mathcal{V} and have the following forms:

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \dots \quad (2.5)$$

and

$$q(z) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \dots, \quad (2.6)$$

respectively. Now, equating coefficients in (2.3) and (2.4), we get

$$(\mu-\lambda)b_0 = (1-\alpha)p_1, \quad (2.7)$$

$$(\mu-2\lambda)(b_1 + (\mu-1)\frac{b_0^2}{2}) = (1-\alpha)p_2, \quad (2.8)$$

$$(\lambda-\mu)b_0 = (1-\alpha)q_1 \quad (2.9)$$

and

$$(2\lambda-\mu)(b_1 - (\mu-1)\frac{b_0^2}{2}) = (1-\alpha)q_2. \quad (2.10)$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \quad (2.11)$$

and

$$b_0^2 = \frac{(1-\alpha)^2(p_1^2 + q_1^2)}{2(\lambda-\mu)^2}. \quad (2.12)$$

Since $\Re\{p(z)\} > 0$ in \mathcal{V} , the function $p(1/z) \in \mathcal{P}$ and hence the coefficients p_n and similarly the coefficients q_n of the function q satisfy the inequality in Lemma 1.3, we get

$$|b_0| \leq \frac{2-2\alpha}{\lambda-\mu}.$$

This gives the bound on $|b_0|$ as asserted in (2.1).

Next, in order to find the bound on $|b_1|$, we use (2.8) and (2.10), which yields,

$$(1-\mu)^2(2\lambda-\mu)^2b_0^4 - 4(1-\alpha)^2p_2q_2 = 4(2\lambda-\mu)^2b_1^2. \quad (2.13)$$

It follows from (2.13) that

$$b_1^2 = \frac{(1-\mu)^2b_0^4}{4} - \frac{(1-\alpha)^2}{(2\lambda-\mu)^2}p_2q_2.$$

Substituting the estimate obtained (2.12), and applying Lemma 1.3 once again for the coefficients p_2 and q_2 , we readily get

$$|b_1| \leq 2(1 - \alpha) \sqrt{\frac{(1 - \mu)^2(1 - \alpha)^2}{(\lambda - \mu)^4} + \frac{1}{(2\lambda - \mu)^2}}.$$

This completes the proof of Theorem 2.1. \square

Next we estimate the coefficients $|b_0|$ and $|b_1|$ for functions in the class $\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda)$.

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the following class:*

$$\tilde{\Sigma}_{\mathcal{M}}^*(\alpha, \mu, \lambda) \quad (0 < \alpha \leq 1; \quad \lambda \geq 1; \quad \mu \geq 0).$$

Then

$$|b_0| \leq \frac{2\alpha}{\lambda - \mu} \quad (2.14)$$

and

$$|b_1| \leq 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1 - \mu)^2}{(\lambda - \mu)^4}}. \quad (2.15)$$

Proof. It follows from (1.8) and (1.9) that

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = [p(z)]^\alpha \quad (2.16)$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = [q(w)]^\alpha, \quad (2.17)$$

where $p(z)$ and $q(w)$ have the forms (2.5) and (2.6), respectively. Now, equating the coefficients in (2.16) and (2.17), we get

$$(\mu - \lambda)b_0 = \alpha p_1, \quad (2.18)$$

$$(\mu - 2\lambda)(b_1 + (\mu - 1)\frac{b_0^2}{2}) = \frac{1}{2} [\alpha(\alpha - 1)p_1^2 + 2\alpha p_2], \quad (2.19)$$

$$-(\lambda - \mu)b_0 = \alpha q_1 \quad (2.20)$$

and

$$(2\lambda - \mu)(b_1 - (\mu - 1)\frac{b_0^2}{2}) = \frac{1}{2} [\alpha(\alpha - 1)q_1^2 + 2\alpha q_2]. \quad (2.21)$$

From (2.18) and (2.20), we find that

$$p_1 = -q_1 \quad (2.22)$$

and

$$b_0^2 = \frac{\alpha^2(p_1^2 + q_1^2)}{2(\lambda - \mu)^2}. \quad (2.23)$$

As discussed in the proof of Theorem 2.1, applying Lemma 1.3 for the coefficients p_2 and q_2 , we immediately have

$$|b_0| \leq \frac{2\alpha}{\lambda - \mu}.$$

This gives the bound on $|b_0|$ as asserted in (2.14).

Next, in order to find the bound on $|b_1|$, by using (2.19) and (2.21), we get

$$2(2\lambda - \mu)^2 b_1^2 + (2\lambda - \mu)^2 (1 - \mu)^2 \frac{b_0^4}{2} = \frac{\alpha^2(\alpha - 1)^2(p_1^4 + q_1^4)}{4} + \alpha^2(p_2^2 + q_2^2) + \alpha^2(\alpha - 1)(p_1^2 p_2 + q_1^2 q_2). \quad (2.24)$$

It follows from (2.24) and (2.23) that

$$\begin{aligned} 2(2\lambda - \mu)^2 b_1^2 &= \frac{\alpha^2(\alpha - 1)^2(p_1^4 + q_1^4)}{4} + \alpha^2(p_2^2 + q_2^2) + \alpha^2(\alpha - 1)(p_1^2 p_2 + q_1^2 q_2) \\ &\quad - \frac{(2\lambda - \mu)^2(1 - \mu)^2 \alpha^4}{8(\mu - \lambda)^4} (p_1^2 + q_1^2)^2. \end{aligned}$$

Applying Lemma 1.3 once again for the coefficients p_1, p_2, q_1 and q_2 , we readily get

$$|b_1| \leq 2\alpha^2 \sqrt{\frac{1}{(2\lambda - \mu)^2} + \frac{(1 - \mu)^2}{(\lambda - \mu)^4}}.$$

This completes the proof of Theorem 2.2. \square

Remark 2.3. For $\lambda = 1$ and $\mu = 0$ the bounds obtained in Theorems 2.1 and 2.2 are coincidence with outcome of [9, Theorem 1 and Theorem 2]. Similarly, various interesting corollaries and consequences could be derived from our results, the details involved may be left to the reader.

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